**Relativistic Bosons**

**Special Relativity for single particle (review)**

So we’ll recall that our attempt at a relativistically invariant theory for bosons (or well, spinless particles) gave us the KG equation, which is as follows, using natural units ℏ = 1, c = 1:



(using the η = (+1, -1, -1, -1) metric) But of course there were issues with this theory. One was that we couldn’t associate with it a positive definite probability density. We didn’t explore this issue per se´, but it also suffers from the same negative energy solutions that the Dirac equation did. Another is that it is incapable of describing particle creation/annihilation. And so we look for a *field* theory of these particles. In this view, the particles will emerge as excitations of the field, in the same way that phonons emerge as excitations of an elastic field, and photons as excitations of the EM field.

**\*real\* bosonic field**

To create a field theory, in absence of any idea of what the field Hamiltonian is per se´, we can turn to many-body theory, and in particular its 2nd quantization formulation. There, the creation/annihilation operators ψ(**x**), ψ†(**x**), develop in time according to the single particle Schrodinger equation [provided no two-body interactions]. So, to turn a single particle theory into a many body theory, one can in effect just convert the wavefunction into an operator. Perhaps the same can be done here? So we would postulate that the bosonic field φ(**x**) develops in time according to the Klein-Gordon equation. We can work backwards and construct the Lagrangian or Hamiltonian, depending on preference, that would produce this evolution. We’d get:

 

(could use time arguments in H too, but it wouldn’t matter as it’s time-independent) From this we can construct the Lagrangian, with these commutation relations:

 

We can verify they’re equivalent by doing,



Note how this H can be interpreted as describing a continuously dense set of ‘harmonic oscillators’. But there is an additional m2φ(x)2 term that wasn’t present in the elastic medium case. So if m were 0, then we’d have an elastic medium type situation. And we’d get an acoustic excitation spectrum. But with non-zero m we’ll get an ‘optical’ like spectrum for our elastic medium. Perhaps the last term can be thought of as a ‘long range’ interaction of some sort. Might passingly note the last two terms actually look like the free energy for magnetic spins. Well let’s get our equation of motion. We’ll do it from the action. We can observe the action is relativistically invariant by integrating L over time, and integrating by parts on and ∇φ, to get:



[here x stands for the 4-vector]



and minimizing,



So our equation of motion for the field is the KG equation again, and that’s what assures us we have the correct H, L. And we’ll couple it with boundary conditions too.



To solve this we can write our field as the inverse spatial Fourier transform of, well, the spatial Fourier transform of our field.



Plugging this in, we find:



The solution to this equation is:



And so our solution is, so far:



We will want to get the properties, commutation relations of the φ0(**k**)’s. So first note that like with the elastic field, since φ(x,t) is Hermitian, we can say:



So we have:



And now the commutation relations between them follow by demanding make φ(x,t) satisfy its canonical commutation relations. Plugging φ(x,t = 0) into



we find:



So we can see that we need,



Now we’ll put φ(x,t) in the free-field expansion. First we’ll note that since φ(x,t) develops harmonically, the ωk are excitations of the field. Therefore the annihilation/creation operators are:



we can work out the normalization by demanding:



Thus our creation/annihilation operators are



And this allows us to write the free field expansion of the field as:



and arrive at finally,



We can plug the FFE into the expression for the Hamiltonian. First note:



And so (can set t = 0 for simplicity here since H is constant)



and,



aaaand,



and now we have, using commutation relations:



which is,



and finally, if we ignore the infinite constant, we’ll get:



The infinity didn’t happen in the discrete case, when we had the elastic field. And formally I suppose this happens because we take V → ∞, which allows us to have arbitrarily large k’s. And so this is a so-called ultraviolet divergence. Arbitrarily large k’s corresponds to arbitrarily small x’s. And so our infinity can be interpreted as a failure of our theory for small x’s. Meaning there is some microscopic structure that exists that we’re failing to model. We can formally get rid of these infinities by normal ordering our H. Let’s note one victory at least, and that is….no more negative energy solutions 😊. And we are obviously able to describe particle annihilation/creation. And the energy spectrum of our particles is what we expect for relativistic particles. Let’s restore units and see what we get. So starting with Ek = √(k2 + m2), the most general result would be something like,



Using some imagination would expedite this process, but if we don’t, then at least we can say that we have a couple requirements. First the two terms in the root must have the same units to be dimensionally consistent. And second these units must be energy2. So, we need,



So we have:



And this is precisely the free particle energy spectrum we know from Special Relativity. So our field theory is doing what we need it to do.

**Example. Ground state wavefunctional?**

What does this look like? Well it should be annihilated by every ak. One way to say this is that Σkak|ψGS> = 0? Or could add up all the occupation numbers and set to zero: Σka†kak|ψGS> = 0. That’s probably what I should do. But if I stick with the former for now, for laziness, we’d say:



Let’s define,



Then we can say,



Then we know how and act on |φ>, so we have:



The solution to this is:



where N is some normalization to be determined, as can be verified by direct differentiation. Well this looks *kind of* like what we’d expect, i.e., harmonic oscillator stuff from before. But it’s probably wrong. Not to mention, the a(x) and b(x) integrals don’t converge. This illustrates difficulty in trying to do this with functional differential equations. Anyway…

**Example. Single Particle States**

An eigenfunction of the Hamiltonian in the truest sense ought to give us the probability that the field assumes a particular configuration, at least if we project the eigenfunction onto ‘field space’. A particular configuration is a function defining the particular eigenvalue that the operator at x, , assumes. Thus it ought to really be a functional – to give the probability of a particular field configuration. In contrast, the eigenfunction for the single dynamical variable should give the probability that it takes on a particular value *x* - this describes a simple function – mapping a number to a number.

In that vein, there are these single particle/multiple particle wavefunctions that bear most resemblance simply to the normal modes, which I will look into below. These clearly are not wavefunctions of the field. They are more like the overlap between an excitation of the field and a field configuration that produces a particle at the position x. Well, since,



And normalized single particle states ought to satisfy,



It would seem that we should say single particle states momentum states are represented by:



Now note that:



Of course this bears resemblance to the wavefunction of a free particle. And it tempts us to conclude that:



which would imply,



This makes sense because then we’d have:



which is self-consistent with what we’d expect to find if we dotted both sides of this equation by <**k**´|. Supposing this to be true, then what do we get if we try?



This is taking a while. But now we have:



which is the correct result (sans a factor of 1/√2! for normalization).

**Conservation Laws**

Let’s just work out some of the conserved quantities, like the the total energy and total momentum.



First:



We already did this one. As for momentum, we’d have:



Filling in the FFE (and setting t = 0 because result is independent of time – could check explicitly but whatever)



Well so we have the usual ∞. But if we ignore the aa and a†a† terms, which would go away when taking any expectation, and flip the minus sign because that doesn’t really matter anyway – I’m guessing the momentum current tensor also comes in with the opposite sign, then we do have what we’d expect for **P**.



FWIW, looks like **℘** and **S** will come out the same.